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Supersymmetry and Integrability in Planar Mechanical Systems

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Abstract We present an $N = 2$-supersymmetric mechanical system whose bosonic sector, with two degrees of freedom, exhibits the most general possible supersymmetric fourth order potential, including the interesting case of SU(2) Yang–Mills theory. The Painlevé test is adopted to discuss integrability and we focus on the rôle of supersymmetry and parity invariance in two space dimensions for the attainment of integrable or non-integrable models, with some remarks on the chaotic behavior. Our result shows that, for the model studied here, the relationships among the parameters, as imposed by supersymmetry, restrict the parameter space in such a way that the reduction on its non-integrable sector is much more severe than on its integrable sector (especially on the non-separable subset of the latter), thus suggesting that supersymmetry may favor (mainly non-separable) integrability.

Keywords Supersymmetric mechanics · Supersymmetry · Integrability · Chaos

1 Introduction

The study of integrability in classical and quantum field theories has been developed for quite a time, actually since the beginning of the eighties, with relevant results that con-

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tributed a great deal for the understanding of these theories and, moreover, allowed the improvement of non-perturbative techniques [1–5]. On the other hand, a number of streams of investigation on chaos has been pushed forward, mainly considering spatially homogeneous field solutions and by performing calculations in the framework of lattice field theory [6–17]. These studies revealed the existence of chaotic solutions in a considerably vast class of gauge theories and, more recently, also in the context of superstrings and supermembrane theories [18–21].

Our work has been motivated by a question stated some years ago [8], which, for the time being, has not been fully answered; namely, whether or not supersymmetry (SUSY) would have a stabilizing rôle for those field theories that, in their non-supersymmetric version, show a chaotic behavior.

In the work of Ref. [22], the authors argue, based on considerations in the framework of Supersymmetric Quantum Mechanics, that an ordered dynamics implies a broken supersymmetry, while exact supersymmetry implies an ergodic dynamics.

$$\begin{array}{cc} \textit{Ordered dynamics} & \textit{Ergodic dynamics} \\ \Downarrow & \Uparrow \\ \textit{Broken SUSY} & \textit{Exact SUSY}. \end{array}$$

It is worthy reminding that an ergodic dynamics is somewhat associated to chaotic dynamics.

Another relevant result in the literature, that establishes a link between chaos and supersymmetry, is found in a work by Horne and Moore [23], where it is stated that the modular space corresponding to the superstring vacuum exhibits chaos. These results enable us to suggest, in the present paper, the hypothesis that a supersymmetric scenario is more viable for the appearance of chaos than a non-supersymmetric framework.

Up to now, a detailed analysis relating supersymmetry and chaos, in much the same way as chaos is studied in field theories, is lacking in the literature. Close to this issue, we should mention a number of attempts to discuss stability and chaos in the framework of brane theories, by concentrating on their bosonic sector [20, 21]. Nevertheless, even in this context, one could put more emphasis on the specific rôle of supersymmetry in the determination of stability and chaos.

A similar situation is observed in connection with the investigation of integrable supersymmetric theories, where the integrable or non-integrable character is ascertained, without however highlighting the mechanisms or those specific properties of supersymmetry which work in favour, or against, integrability [24–32].

Our work sets out to tackle this issue, that we believe should be more manifestly worked out. To pursue an investigation focusing on the rôle of supersymmetry in connection with integrability and chaos, we propose to start off from a supersymmetric mechanical system, rather than a field-theoretic model. The system we choose to work with is built up as the $N = 2$-extended supersymmetric version of a dimensionally reduced $SU(2)$ Yang–Mills (YM) theory that arises when spatially homogeneous fields are considered and a particular ansatz on the gauge potentials is adopted in the dimensional reduction scheme so that only two degrees of freedom survive [33] in the mechanical limit. We also devote special attention to the rôle of parity symmetry, since we assume the latter is an invariance of the interactions involved in the systems we shall be considering. Our analysis of integrability shall therefore rely on our considerations on supersymmetry and parity invariance. They dictate special conditions in the space of parameters so that, instead of having to take by decree special choices of these parameters, as it is usually done, we invoke these two invariances to naturally restrict and select possibilities in parameter space. As a matter of fact, we anticipate that parity may appear in two versions for planar systems, and this point shall be suitably taken care of here.

Our paper is organized as follows. In Sect. 2, we propose a general 2-dimensional purely bosonic model with parity symmetry and we identify the cases of integrability. Next, the $N = 2$-supersymmetric extension of the model is written down in Sect. 3. The complete bosonic sector, now enlarged by the presence of two supersymmetries, is discussed in full details in Sect. 4, where we pay due attention to the rôle of parity and we pick out Painlevé test as a criterium to infer about integrability. In Sect. 5, we reassess the question of the integrability for the bosonic sector of our $N = 2$-model, but now taking into account the constraints dictated by parity whenever it is imposed also to the fermionic interactions. A very restrictive class of potentials comes out that fulfills integrability. In Sect. 6, we perform a brief chaos analysis and, finally, in Sect. 7, we present our Final Discussions and we draw our General Conclusions.

2 The Ordinary Bosonic Model with Considerations on Parity Symmetry

We assume the most general fourth-order polynomial potential for two degrees of freedom described by the variables x and y:

$$V = C_1x^4 + C_2y^4 + C_3x^3y + C_4xy^3 + C_5x^2y^2 + C_6x^3 + C_7y^3 + C_8x^2y + C_9xy^2 + C_{10}x^2 + C_{11}y^2, \tag{1}$$

where the term in xy was not considered, since it may be canceled out by means of a proper linear transformation (a rotation in the x–y plane).

It may be considered as a sort of protopotential used to build up a general non-supersymmetric polynomial potential up to fourth order. We are bound to fourth order because we have in mind mechanical models derived from Yang–Mills field theories and these, as we know, display self-interaction vertices for three and four potentials (or coordinates, in the mechanical version). Since we are interested in realistic models, we impose parity symmetry which is respected by mechanical, electromagnetic and strong-force models. We shall not be dealing with models coming from chiral gauge theories.

To implement parity in the model, we have to consider that there are two possibilities, since we are in a two-dimensional space:

$$x\text{-parity:} \quad \begin{cases} x \to -x \\ y \to y, \end{cases} \tag{2}$$

or

$$y\text{-parity:} \quad \begin{cases} x \to x \\ y \to -y. \end{cases} \tag{3}$$

In the first case, the resulting potential is

$$V = C_1x^4 + C_2y^4 + C_5x^2y^2 + C_7y^3 + C_8x^2y + C_{10}x^2 + C_{11}y^2. \tag{4}$$

This potential looks like the sum of two well-known potentials:

- A quartic potential (Yang–Mills-type)

$$V_{\text{YM}} = Ax^2 + By^2 + ax^4 + by^4 + dx^2y^2, \tag{5}$$

Table 1 Some known integrable cases of YM-type potentials

Notation of (5)	Notation of (1)
(a) $A=B,\ a=b,\ d=6a$	$C_{10}=C_{11},\ C_1=C_2,\ C_5=6C_1$
(b) $A,\ B,\ a=b,\ d=2a$	$C_{10},\ C_{11},\ C_1=C_2,\ C_5=2C_1$
(c) $A=4B,\ a=16b,\ d=12a$	$C_{10}=4C_{11},\ C_1=16C_2,\ C_5=12C_1$
(d) $A=4B,\ a=8b,\ d=6b$	$C_{10}=C_{11},\ C_1=C_2,\ C_5=6C_1$
(e) $d=0$ (trivial)	$C_5=0$

Table 2 Some known integrable cases for the Henon–Heiles potential

Notation of (6)	Notation of (1)
(a) $M=N,\ m=-n$	$C_{10}=C_{11},\ C_7=\frac{1}{3}C_8$
(b) $M,\ N,\ 6m=-n$	$C_{10},\ C_{11},\ C_7=2C_8$
(c) $M=16N,\ 16m=-n$	$C_{10}=16C_{11},\ C_7=\frac{16}{3}C_8$
(d) $m=0$ (trivial)	$C_8=0$

which is known to be integrable in the cases [34] shown in Table 1. When A, B and d are the only non-vanishing parameters, this potential is called "Yang–Mills-Higgs", while the case in which only the parameter d is non-vanishing is usually called "pure Yang–Mills" [33].

- The Henon–Heiles (HH) potential

$$V_{\mathrm{HH}}=\frac{1}{2}(Mx^2+Ny^2)+mx^2y-\frac{n}{3}y^3, \tag{6}$$

that exhibits well-known integrable cases [34] shown in Table 2.

For the case of y-parity, similar conclusions may be drawn, with appropriate exchanges of coordinate and constant labels.

3 The supersymmetric model

Now, we shall consider an $N=2$-supersymmetric mechanical model [35], defined as follows. The two Grassmannian parameters of the superspace will be denoted by θ and $\bar{\theta}$. The two real Cartesian coordinates of a planar particle, x and y, are the bosonic components of the superfield coordinates, which are given by

$$X(t,\theta,\bar{\theta})=x(t)+\Theta^{\dagger}\gamma_1\Lambda(t)+\Lambda^{\dagger}(t)\gamma_1\Theta-\frac{1}{2}\Theta^{\dagger}\gamma_3\Theta f_1(t) \tag{7}$$

and

$$Y(t,\theta,\bar{\theta})=y(t)+\Theta^{\dagger}\gamma_2\Xi(t)+\Xi^{\dagger}(t)\gamma_2\Theta-\frac{1}{2}\Theta^{\dagger}\gamma_3\Theta f_2(t), \tag{8}$$

with:

$$\Theta\equiv\begin{pmatrix}\theta\\ \bar{\theta}\end{pmatrix},\qquad \Lambda\equiv\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix},\qquad \Xi\equiv\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{9}$$

where all the λ's and ξ's are Grassmannian variables. The γ_j's are the Dirac matrices corresponding to the two-dimensional Euclidean space under consideration and they may be chosen so as to coincide with the Pauli matrices: $\gamma_i \equiv \sigma_i$ and $\gamma_3 \equiv -i\gamma_1\gamma_2 = \sigma_3$. Θ is a Majorana spinor, which, in this particular representation of the γ-matrices, takes the form given in (9), where the "bar" stands for complex conjugation. On the other hand, Λ and Ξ are Dirac fermions. Therefore, (7–8) yield:

$$X = x + \theta\psi_1 - \bar{\theta}\bar{\psi}_1 + \theta\bar{\theta} f_1 \tag{10}$$

and

$$Y = y + \theta\psi_2 - \bar{\theta}\bar{\psi}_2 + \theta\bar{\theta} f_2, \tag{11}$$

where it is noteworthy to remark that it is precisely the combination $\psi_1 \equiv (\lambda_1 - \bar{\lambda}_2)$, along with its complex conjugate $\bar{\psi}_1 \equiv (\bar{\lambda}_1 - \lambda_2)$, that carry the fermionic degrees of freedom of X. Similarly, the spinorial degrees of freedom of Y are all contained in $\psi_2 \equiv i(\xi_1 - \bar{\xi}_2)$ and its complex conjugate, $\bar{\psi}_2 \equiv -i(\bar{\xi}_1 - \xi_2)$.

The supersymmetry covariant derivatives are given by

$$D \equiv \partial_{\bar{\theta}} - i\theta\partial_t, \tag{12}$$

$$\bar{D} \equiv \partial_{\theta} - i\bar{\theta}\partial_t \tag{13}$$

and satisfy:

$$D^2 = 0, \tag{14}$$

$$\bar{D}^2 = 0, \tag{15}$$

$$\{D, \bar{D}\} = -2i\partial_t. \tag{16}$$

The super-action to be considered contains, besides the kinetic terms, the most general superpotential, up to third order in the superfield coordinates (this implies a fourth-order potential in terms of the physical coordinates),

$$S = \int dt d\theta d\bar{\theta} \left\{ \frac{M}{2}[DX\bar{D}X + DY\bar{D}Y] + U(X, Y) \right\}, \tag{17}$$

where the first term gives rise to the kinetic terms and the superpotential $U(X, Y)$ is assumed to be given by:

$$U(X, Y) = k_1 X^2 Y + k_2 XY^2 + k_3 X^2 + k_4 Y^2 + k_1' X^3 + k_2' Y^3, \tag{18}$$

the k's being arbitrary real constants. The term in XY was not considered for the same reason stated after (1), now referring to a rotation in the plane of the superfield coordinates, X and Y. With regard to the terms linear in X and in Y, they were not included, since, similarly to what happens with the terms linear in x and in y in usual mechanics, their coefficients are the Euler–Lagrange equations. The equations of motion may be used to eliminate the non-dynamical degrees of freedom, f_j, and, thus, the super-action, $S = \int dt L$, yields the following Lagrangian, which is therefore the most general $N = 2$-supersymmetric Lagrangian with a fourth-order potential (summation over repeated indices assumed):

$$L = \frac{M\dot{x}^2}{2} + i\frac{M}{2}(\bar{\psi}_j\dot{\psi}_j + \psi_j\dot{\bar{\psi}}_j) - \frac{k_1^2 + 9k_1'^2}{2M}x^4 - \frac{k_2^2 + 9k_2'^2}{2M}y^4 - \frac{6k_1k_1' + 2k_1k_2}{M}x^3y$$
$$- \frac{6k_2k_2' + 2k_1k_2}{M}xy^3 - \frac{2k_1^2 + 2k_2^2 + 3k_1'k_2 + 3k_1k_2'}{M}x^2y^2 - \frac{6k_3k_1'}{M}x^3 - \frac{6k_4k_2'}{M}y^3$$
$$- \frac{4k_1k_3 + 2k_1k_4}{M}x^2y - \frac{4k_2k_4 + 2k_2k_3}{M}xy^2 - \frac{2k_3^2}{M}x^2 - \frac{2k_4^2}{M}y^2$$
$$- 2[k_1(\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2) + k_2\psi_2\bar{\psi}_2 + 3k_1'\psi_1\bar{\psi}_1]x$$
$$- 2[k_2(\psi_1\bar{\psi}_2 - \bar{\psi}_1\psi_2) + k_1\psi_1\bar{\psi}_1 + 3k_2'\psi_2\bar{\psi}_2]y$$
$$- 2k_3\psi_1\bar{\psi}_1 - 2k_4\psi_2\bar{\psi}_2. \tag{19}$$

All the SUSY technicalities are over and, in the next sections, the integrability conditions for the Lagrangian above will be discussed, and the influence of supersymmetry and parity invariance shall be highlighted.

4 The Bosonic Sector and its Integrability

The application of the Painlevé test (for a short review, see Appendix) directly to the bosonic sector is not actually a good procedure, for the resolution of the systems that appear in the analysis becomes very complex.

In this section, we shall take into consideration the observation that the original model is not invariant under the two classes of parity transformations. This may set a more formal framework.

So, in a first attempt, we will impose parity symmetry only to the bosonic sector of the theory and, after that, we shall check how the constraints imposed by this invariance affect the integrability of the model.

Adopting invariance under x-parity, we have the following constraints on the bosonic part of the potential that appears in (19):

$$\frac{6k_1k_1' + 2k_1k_2}{M} = 0, \tag{20}$$

$$\frac{6k_2k_2' + 2k_1k_2}{M} = 0, \tag{21}$$

$$\frac{6k_3k_1'}{M} = 0, \tag{22}$$

$$\frac{4k_2k_4 + 2k_2k_3}{M} = 0. \tag{23}$$

4.1 Parameters surviving the parity constraints

Solving the system of conditions for $k_1, k_2, k_1', k_2', k_3$ and k_4, we obtain as solutions the following possibilities (the parameters shown between commas in each case below may assume any value, while those not appearing are all vanishing):

$$\text{1st case:} \quad \{k_1', k_2', k_4\}; \tag{24}$$

$$\text{2nd case:} \quad \{k_1, k_2', k_4, k_3\}; \tag{25}$$

$$\text{3rd case:} \quad \{k_1', k_2\}; \tag{26}$$

$$\text{4th case:} \quad \{k_1', k_1, k_2 = -3k_1', k_2' = -k_1/3\}; \tag{27}$$

$$\text{5th case:} \quad \{k_2, k_3, k_4 = -k_3/2\}. \tag{28}$$

To study the consequences of these solutions we shall present in the next subsection the Painlevé test (see Appendix), which has been very used in the search for integrable systems, for being an algorithm.

4.2 Applying the Painlevé test

1st case

For the first case, $\{k_1', k_2', k_4\}$, we have the following potential:

$$Pot_1 = \frac{9k_1'^2}{2M}x^4 + \frac{9k_2'^2}{2M}y^4 + \frac{6k_4k_2'}{M}y^3 + \frac{2k_4^2}{M}y^2, \tag{29}$$

which represents two uncoupled degrees of freedom and thus is integrable. Indeed, applying the Painlevé test, we obtain four branches referring to these uncoupled systems and which survive the test.

2nd case

For the second case, $\{k_1, k_2', k_4, k_3\}$, we have the following potential:

$$Pot_2 = \frac{k_1^2}{2M}x^4 + \frac{9k_2'^2}{2M}y^4 + \frac{2k_1^2 + 3k_1k_2'}{M}x^2y^2 + \frac{6k_4k_2'}{M}y^3 + \frac{4k_1k_3 + 2k_1k_4}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_4^2}{M}y^2, \tag{30}$$

with dominant potencies:

$$\alpha_1 = -1, \qquad \alpha_2 = -1 \tag{31}$$

and four branches with the following expressions for the resonances:

$$-1, \quad 4, \quad \frac{2k_1 - 3k_2'}{k_1}, \quad \frac{3k_2' + k_1}{k_1}, \tag{32}$$

that will show integer resonances if we set $k_2' = \frac{n}{3}k_1$, where $n = \{-1, 0, 1, 2\}$.

For the case $n = -1$, it is not possible to determine the resonances.

For the case $n = 0$, we have the following potential:

$$Pot_3 = \frac{k_1^2}{2M}x^4 + \frac{2k_1^2}{M}x^2y^2 + \frac{4k_1k_3 + 2k_1k_4}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_4^2}{M}y^2. \tag{33}$$

It does not pass the Painlevé test in the sense that there appears a compatibility condition that cannot be fulfilled:

$$-4i\sqrt{2}(18k_1^2x_1^2 - 5k_4^2 - 4k_3k_4) = 0, \tag{34}$$

except for the following trivial cases: if k_1, k_3 and $k_4 = 0$, which cancels out the potential; and if $k_1 = 0$ and $k_4 = -\frac{4}{5}k_3$, which leads to the harmonic potential

$$Pot_4 = \frac{2k_3^2}{M}x^2 + \frac{32k_3^2}{25M}y^2 \tag{35}$$

and therefore constitutes a trivial integrable case.

For the case $n = 1$, we have the following potential:

$$\begin{aligned} Pot_5 = {} & \frac{k_1^2}{2M}x^4 + \frac{k_1^2}{2M}y^4 + \frac{3k_1^2}{M}x^2y^2 + \frac{2k_4k_1}{M}y^3 \\ & + \frac{4k_1k_3 + 2k_4k_1}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_4^2}{M}y^2 \end{aligned} \tag{36}$$

and, now, we obtain four branches with the following resonances:

$$-1, \quad 1, \quad 2, \quad 4, \tag{37}$$

but with the following compatibility condition:

$$-2(-k_4 + k_3)M = 0, \tag{38}$$

to be verified in the resonance $j = 1$ of the first and of the second branches. Setting $k_3 = k_4$, the potential is now written as below:

$$\begin{aligned} Pot_6 = {} & \frac{k_1^2}{2M}x^4 + \frac{k_1^2}{2M}y^4 + \frac{3k_1^2}{M}x^2y^2 + \frac{2k_3k_1}{M}y^3 \\ & + \frac{6k_3k_1}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_3^2}{M}y^2, \end{aligned} \tag{39}$$

which is the sum of the two integrable cases shown in Table 1, item (a), and Table 2, item (a). Thus, the system is expected to be integrable. Indeed, it passes the Painlevé test, with dominant potencies:

$$\alpha_1 = -1, \qquad \alpha_2 = -1. \tag{40}$$

The values of the resonances for the two branches are:

$$-1, \quad 1, \quad 2, \quad 4, \tag{41}$$

and for the first branch the coefficients of the dominant terms are:

$$x_0 = \frac{iM}{2k_1}, \qquad y_0 = \frac{iM}{2k_1}. \tag{42}$$

For the second branch, the coefficients read as follows:

$$x_0 = -\frac{iM}{2k_1}, \qquad y_0 = -\frac{iM}{2k_1}. \tag{43}$$

In the first branch, the arbitrary coefficients are:

$$y_1, \quad y_2 \quad \text{and} \quad y_4, \tag{44}$$

and the arbitrary coefficients of the second branch are:

$$y_1, \quad y_2 \quad \text{and} \quad x_4. \tag{45}$$

Reminding that the variable t_0 is the fourth arbitrary quantity corresponding to the resonance -1, thus, this (fourth order) system possesses four arbitrary coefficients and, therefore, it is integrable.

For the case $n = 2$, we have the following potential:

$$\begin{aligned} Pot_7 = {} & \frac{k_1^2}{2M}x^4 + \frac{2k_1^2}{M}y^4 + \frac{4k_1^2}{M}x^2y^2 + \frac{4k_4k_1}{M}y^3 \\ & + \frac{4k_1k_3 + 2k_4k_1}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_4^2}{M}y^2. \end{aligned} \tag{46}$$

It was not possible to determine the dominant terms.

3rd case

For the third case, $\{k_1', k_2\}$, we have the following potential:

$$Pot_8 = \frac{9k_1'^2}{2M}x^4 + \frac{k_2^2}{2M}y^4 + \frac{2k_2^2 + 3k_1'k_2}{M}x^2y^2. \tag{47}$$

The expressions for the resonances in this case are:

$$-1, \quad 4, \quad \frac{3k_1' + k_2}{k_2}, \quad \frac{2k_2 - 3k_1'}{k_2}, \tag{48}$$

that will show integer resonances if we set $k_1' = \frac{n}{3}k_2$, where $n = \{-1, 0, 1, 2\}$.

For the case $n = -1$, the potential is

$$Pot_9 = \frac{k_2^2}{2M}x^4 + \frac{k_2^2}{2M}y^4 + \frac{k_2^2}{M}x^2y^2, \tag{49}$$

and the system passes the Painlevé test, as expected, since this is a known integrable case already shown in item (b) of Table 1.

For the case $n = 0$, the system does not pass the test with the following potential:

$$Pot_{10} = \frac{k_2^2}{2M}y^4 + \frac{2k_2^2}{M}x^2y^2, \tag{50}$$

because there appears the following compatibility condition:

$$18ik_2^2y_1^2 = 0, \tag{51}$$

which is satisfied only if k_2 is equal to zero, but in this case the potential vanishes.

For the case $n = 1$, the potential is

$$Pot_{11} = \frac{k_2^2}{2M}x^4 + \frac{k_2^2}{2M}y^4 + \frac{3k_2^2}{M}x^2y^2, \tag{52}$$

and the system passes the Painlevé test, as expected, since this is a known integrable case already shown in item (a) of Table 1.

For the case $n=2$, the system does not pass the test with the following potential:

$$Pot_{12}=\frac{2k_2^2}{M}x^4+\frac{k_2^2}{2M}y^4+\frac{4k_2^2}{M}x^2y^2, \tag{53}$$

because it was not possible to determine the dominant terms.

4th case

For the fourth case, $\{k_1', k_1, k_2=-3k_1', k_2'=-k_1/3\}$, we have the following potential (quartic):

$$Pot_{13}=\frac{5k_1^2}{M}x^4+\frac{5k_1^2}{M}y^4+\frac{10k_1^2}{M}x^2y^2, \tag{54}$$

which is the same as (49), already shown to be integrable.

5th case

For the fifth case, $\{k_2, k_3, k_4=-k_3/2\}$, we have the following potential:

$$Pot_{14}=\frac{k_2^2}{2M}y^4+\frac{2k_2^2}{M}x^2y^2+\frac{2k_3^2}{M}x^2+\frac{k_3^2}{2M}y^2. \tag{55}$$

This potential does not pass in the Painlevé test because the following compatibility condition appears:

$$-3ik_3^2+18ik_2^2y_1^2=0, \tag{56}$$

which is satisfied only if $k_2=k_3=0$, and this eliminates our potential.

Summary

In Table 3, we summarize our results of this section, in which some integrable potentials were found by means of the application of the Painlevé test to that cases of the bosonic sector of the Lagrangian (19) which preserve x-parity (similar results may immediately be obtained for the y-parity, by proper exchanges in the coordinates and coefficients labels). In the next section, the x-parity will be imposed also for the fermionic sector and we will conclude that potentials Pot_9 and Pot_{11} are the only ones also compatible with this requirement (as well as with regard to the y-parity).

5 The Integrability of the Bosonic Sector with Parity Considerations for the Complete Model

As verified in the previous section, by imposing parity to the bosonic sector, the task of finding integrable cases became less arbitrary, in that the choice of the coefficients in the terms of the potential was guided by the argument of parity invariance. In spite of that, it was still necessary to fix by hand the values of some parameters when applying Painlevé test to recover the integrable cases we have listed previously.

In this section, we shall impose the parity symmetry not only to the bosonic sector but also to the fermionic interactions, and we shall verify to which extent the resulting constraints on the parameters are compatible with integrability.

Table 3 Integrable cases found with the Painlevé test for the cases of the bosonic part of the Lagrangian (19) which are x-parity preserving

Case	Potential	Comments
1st	Pot_1, (29)	Two uncoupled degrees of freedom
2nd	Pot_4, (35)	A harmonic potential, trivially integrable
	Pot_6, (39)	YM- plus HH-type; Table 1(a) + Table 2(a)
3rd	Pot_9, (49)	YM-type, Table 1(b); also compatible with fermionic x- and y-parity symmetry (see next section)
	Pot_{11}, (52)	YM-type, Table 1(a); also compatible with fermionic x- and y-parity symmetry (see next section)
4th	$Pot_{13} = Pot_9$	–
5th	None was found	–

5.1 Two-Component Formulation of the Fermionic Sector

Since the model is classic and non-relativistic, and defined in a two-dimensional Euclidean space, E^2, the covariance group is SO(2). We adopt the representation below for the Clifford algebra:

$$\gamma_1 = \sigma_x, \tag{57}$$

$$\gamma_2 = \sigma_y, \tag{58}$$

$$\gamma_3 = -i\gamma_1\gamma_2 = \sigma_z, \tag{59}$$

such that:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad j = 1, 2, \tag{60}$$

$$\{\gamma_i, \gamma_3\} = 0. \tag{61}$$

For a general spinor,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{62}$$

the action of SO(2) is given by

$$\Psi' = e^{-\frac{i}{2}\omega\sigma_z}\Psi, \tag{63}$$

where ω is the rotation angle; therefore $\Psi^\dagger\Psi$ is invariant.

Now, we try to identify x- and y-parities in the spinorial space. To do that, we start off from the Dirac equation:

$$i\gamma_1\partial_x\Psi + i\gamma_2\partial_y\Psi = 0, \tag{64}$$

to which we impose x-parity symmetry:

$$\Psi(t;\vec{x}) \xrightarrow{P} \Psi'(t';\vec{x}') = R\Psi(t;\vec{x}) = R\Psi(t';-x',y'), \tag{65}$$

where R represents the parity matrix in the spinor space:

$$\gamma_1 R = -R\gamma_1, \tag{66}$$

$$\gamma_2 R = R\gamma_2. \tag{67}$$

Then, our parity matrix may be chosen as

$$R = \gamma_2 \tag{68}$$

and, thus,

$$\Psi'(t'; \vec{x}') = \gamma_2 \Psi(t; \vec{x}). \tag{69}$$

So, all spinors, up to a phase factor, transform under parity by means of the γ_2-matrix.

Considering the other possibility, that is, the y-parity,

$$P \begin{cases} x \;\to\; x, \\ y \;\to\; -y, \end{cases} \tag{70}$$

one can readily check that this parity is represented by the γ_1-matrix:

$$\begin{aligned} \Psi \;&\to\; \gamma_1 \Psi, \\ \Psi'(t'; \vec{x}') \;&\to\; \gamma_1 \Psi(t; \vec{x}). \end{aligned} \tag{71}$$

5.2 The Integrability with the Parity Constraints from the Fermionic Sector

To include the constraints dictated by x- or y-parity symmetry for the complete (bosonic + fermionic) model, we propose to actually carry out the analysis directly in terms of the superfields (7) and (8). Rather than following the lengthy procedure of considering all the terms of the component-field action, we propose to work without quitting superspace.

The action of the x-parity on the superfields is given by

$$X \;\to\; -X \quad \text{and} \quad Y \;\to\; Y, \tag{72}$$

provided that

$$\begin{aligned} \Theta \;&\to\; \gamma_2 \Theta, \\ \Lambda \;&\to\; \gamma_2 \Lambda, \\ \Xi \;&\to\; -\gamma_2 \Xi, \\ f_1 &\to\; f_1, \\ f_2 &\to\; -f_2. \end{aligned}$$

With these parity assignments to the fermions and auxiliary fields, the superfield coordinates transform under parity exactly as above. Moreover, by virtue of the specific choice of γ_2, we have that parity acts on $d\theta$, $d\bar{\theta}$ and the covariant derivatives as below:

$$\begin{aligned} D \;&\to\; -i\bar{D}, \qquad \bar{D} \to iD; \\ d\theta \;&\to\; i d\bar{\theta}, \qquad d\bar{\theta} \to -i d\theta. \end{aligned}$$

With all the prescriptions above, the volume element, $dtd\theta d\bar{\theta}$, picks a minus sign. This means that the kinetic terms are naturally invariant, but parity symmetry of the potential sets

$$k_1 = k_3 = k_4 = k_2' = 0, \tag{73}$$

with k_2 and k_1' non-vanishing.

These parameter constraints are the same as the "3rd case" of Sect. 4.2, which was obtained when only the bosonic sector was considered and for which we found only two integrable cases: Potentials 9 and 11, that we rename now as below:

$$Pot_{\text{susy}1-x} = \frac{k_2^2}{2M}x^4 + \frac{k_2^2}{2M}y^4 + \frac{k_2^2}{M}x^2y^2, \tag{74}$$

and

$$Pot_{\text{susy}2-x} = \frac{k_2^2}{2M}x^4 + \frac{k_2^2}{2M}y^4 + \frac{3k_2^2}{M}x^2y^2. \tag{75}$$

So, from all integrable cases found when we considered only the bosonic sector, only the two potentials above preserve x-parity under consideration of the complete model.

On the other hand, if we contemplate y-parity symmetry for the whole model, we have

$$X \to X \quad \text{and} \quad Y \to -Y, \tag{76}$$

provided that

$$\begin{aligned} \varTheta &\to \gamma_1\varTheta, \\ \varLambda &\to -\gamma_1\varLambda, \\ \varXi &\to \gamma_1\varXi, \\ f_1 &\to -f_1, \\ f_2 &\to f_2. \end{aligned}$$

Also, $D \to -i\bar{D}$, $\bar{D} \to iD$, $d\theta \to id\bar{\theta}$ and $d\bar{\theta} \to -id\theta$.

So, as in previous case, y-parity invariance is ensured only for those superfield monomials that change sign under parity. This then impose:

$$k_2 = k_3 = k_4 = k_1' = 0, \tag{77}$$

while k_1 and k_2' are the only coefficients compatible with y-parity invariance.

These constraints on the parameters correspond to only one set of solutions that is found when only the bosonic sector is considered in connection with the y-parity, in a similar way to what happens for x-parity. There are only two integrable cases that we shall present below:

$$Pot_{\text{susy}1-y} = \frac{k_1^2}{2M}x^4 + \frac{k_1^2}{2M}y^4 + \frac{k_1^2}{M}x^2y^2, \tag{78}$$

and

$$Pot_{\text{susy}2-y} = \frac{k_1^2}{2M}x^4 + \frac{k_1^2}{2M}y^4 + \frac{3k_1^2}{M}x^2y^2. \tag{79}$$

So, from all integrable cases found when only the bosonic sector is considered, only the two potentials above preserve y-parity if the whole model is analyzed. Now, one immediately notes that the potentials (78) and (79) are the same as (74) and (75) (since the label of the parameter, k_1 or k_2, is not relevant), thus leading us to the conclusion that the requirement of either x- or y-parity symmetry in the whole model implies the same two possibilities for the potential in order to the system be integrable (under the criterium of Painlevé test).

6 Chaotic Behavior

In the previous sections, we carried out an analysis of the integrability of the bosonic sector of supersymmetric models and we have pointed out the appearance of integrable cases for both coupled and non-coupled systems. The coupled cases result to be classified into two types: those with a quartic potential and those whose potential is functionally the superposition of a quartic and a Henon–Heiles potential.

Taking the same two types of potentials, but, now, considering the non-integrable cases, we can conclude that, for the cases where the potentials have a quartic form, there is no need to go through a chaos analysis, for this issue has already been discussed in the literature we have previously referred to.

On the other hand, the study of chaos for the cases in which the potential is the superposition of quartic and Henon–Heiles terms is more complex and deserves a separate work. However, in this section, we shall give an example to illustrate how this type of non-integrable potential admits order-chaos transition. Consider potential number 7 of Sect. 4.2:

$$Pot_7 = \frac{k_1^2}{2M}x^4 + \frac{2k_1^2}{M}y^4 + \frac{4k_1^2}{M}x^2y^2 + \frac{4k_4k_1}{M}y^3 + \frac{4k_1k_3 + 2k_4k_1}{M}x^2y + \frac{2k_3^2}{M}x^2 + \frac{2k_4^2}{M}y^2 \tag{80}$$

and let us make use of the following techniques: Lyapunov characteristic exponent (LCE) [36, 37], phase portraits and Poincaré sections [38]. The Lyapunov exponent is a useful tool to quantify the divergence or convergence of initial nearby trajectories for a dynamical system. In a chaotic system, there is at least one positive Lyapunov exponent, defined as

$$\sigma_i = \lim_{t\to\infty} \ln \frac{d_i(t)}{d_i(0)}, \tag{81}$$

where $d_i(t)$ is a deformation measure of the small hypersphere of initial conditions in the phase space along the trajectory. The asymptotic rate of expansion of the largest axis is given by the largest LCE. By phase portrait we mean a graph of the dynamical variables in phase space that is used to provide a qualitative insight of the dynamical behavior of the system under study. Similarly, the Poincaré section [38, 39] (a plot generated by the points arising from the flux of the differential system intersecting a plane in phase space), yields a qualitative information on the dynamical behavior of the system. The accuracy of our computation was verified by checking if the Hamiltonian was conserved during the simulation.

Fixing $k_1 = 10$ and $M = k_3 = k_4 = 1$, the potential acquires the following form:

$$V = 50x^4 + 200y^4 + 400x^2y^2 + 40y^3 + 60x^2y + 2x^2 + 2y^2. \tag{82}$$

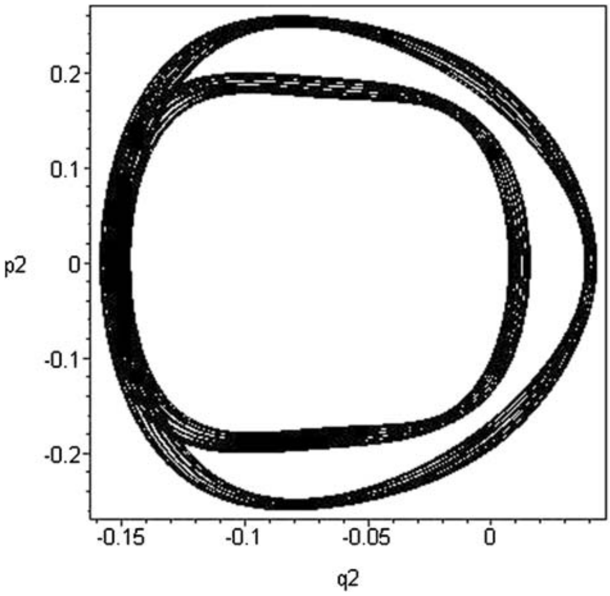

Fig. 1 Phase portrait for the first case considered (see text)

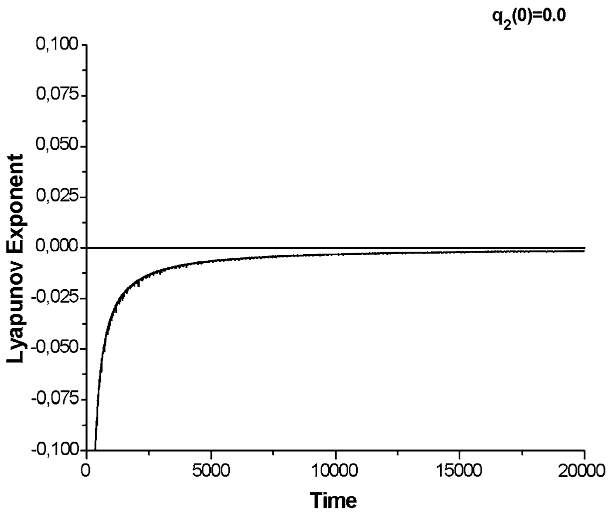

Fig. 2 Lyapunov exponent for the first case considered (see text)

We calculate the largest σ_i and its respective phase portraits and we present two cases for the same set of parameters fixed above, but with different initial conditions. First, with $p_1(0) = 0.1$, $p_2(0) = 0.1$, $q_1(0) = 0.1$, $q_2(0) = 0.0$, Energy $= 0.035$; it presents regular behavior (see Figs. 1, 2 and 3). Here and in the figures, q_i stand for the coordinates x and y, and p_i are the associated momenta.

The second case is given by: $p_1(0) = 0.1$, $p_2(0) = 0.1$, $q_1(0) = 0.1$, $q_2(0) = 0.18$, Energy $= 0.780631$; it presents chaotic behavior (see Figs. 4, 5 and 6).

In the case in which the selected initial conditions correspond to the energy value 0.035, the system exhibits regular behavior; this can be seen in the corresponding phase portrait and Lyapunov exponent graphics. For the case in which the chosen initial conditions correspond

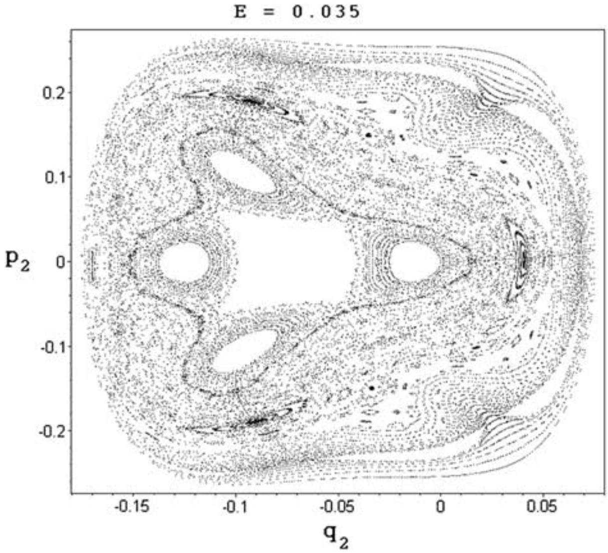

Fig. 3 Poincaré section in the plane $p_1 = 0$ for an energy value of 0.035

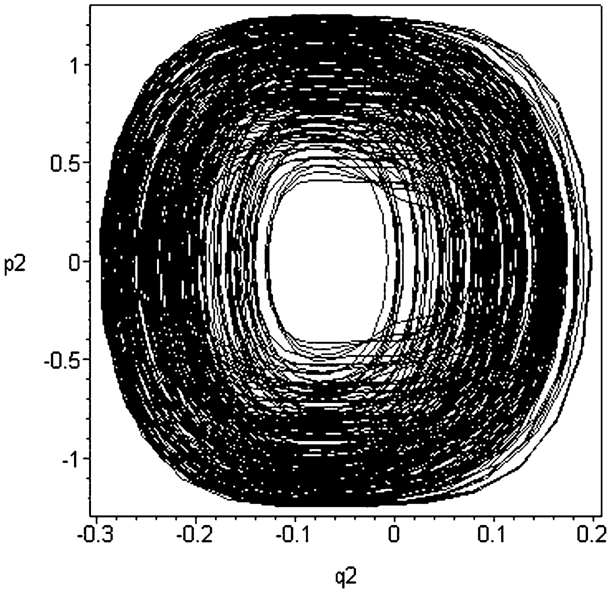

Fig. 4 Phase portrait for the second case considered (see text)

to the energy value 0.780631, the system exhibits chaotic behavior, as it may be seen by the corresponding figures.

However, a global view of the dynamics may be obtained by means of the corresponding Poincaré sections. The Poincaré section for the energy value 0.035 (Fig. 3) shows, in its most peripheral part, various separatrices and, in its interior, a chaotic region with basins and various regularity islands. On the other hand, for the Poincaré section corresponding to the energy value 0.780631 (Fig. 6), one sees a dense region of regularity in the center and the most part of the volume characterized by a chaotic behavior, where three regularity

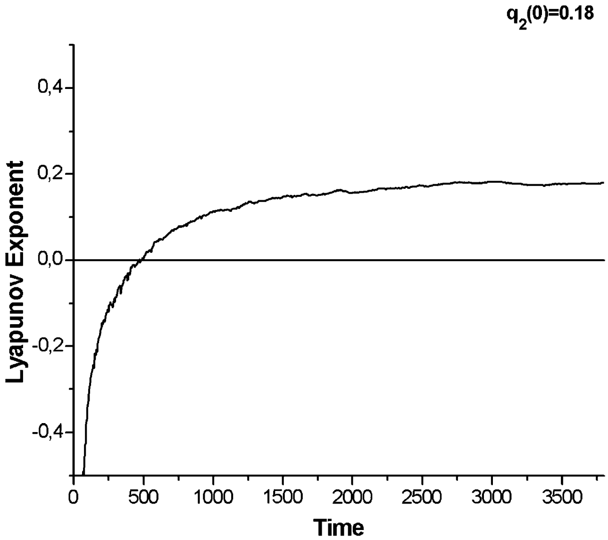

Fig. 5 Lyapunov exponent for the second case considered (see text)

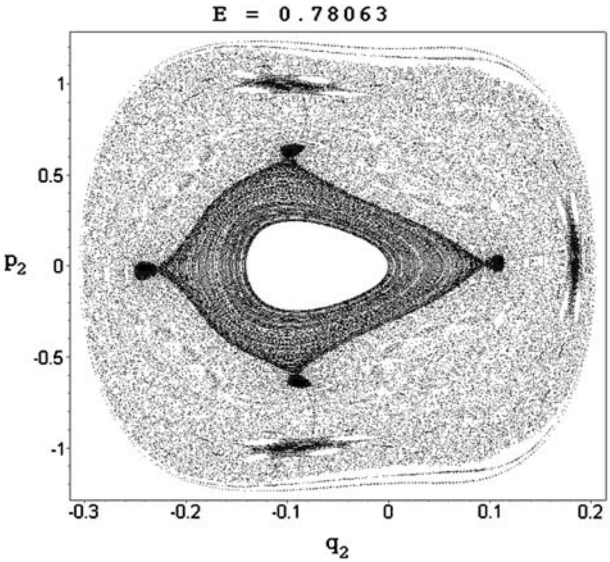

Fig. 6 Poincaré section in the plane $p_1 = 0$ for an energy value of 0.780631

islands may be distinguished. Another apparent characteristic is the reduction in the number of separatrices.

These examples show that this non-integrable system exhibits both ordered and chaotic dynamics. This is not a new result in the case of Hamiltonian systems, where order and chaos may coexist for a given value of the energy. However, this does not set up a rule. The mechanical version of the pure Yang–Mills case mentioned in Sect. 2 exhibits only chaos. Similarly, there are also other non-integrable models that do not exhibit chaos. Since, besides the integrability analysis, the central question in this work is the identification of chaos in supersymmetric theories, the simple demonstration of its existence constitutes the most important aspect, as compared to a detailed study of chaos in these models, which, as already mentioned, deserves a separate work.

7 Conclusions

The full interplay between SUSY and integrability in mechanical systems is not yet known, although some works in this direction do exist, among which we may cite Refs. [24–32]. By "full" we mean what follows below.

Given a purely bosonic, non-supersymmetric potential whose possible relations between its parameters may turn it integrable or not, the implementation of SUSY will certainly restrict the parameter space to a subset of the original one. Whether or not this restriction will be more severe on the integrable or on the non-integrable sector of the original parameter space is an interesting question that, as far as we know, has not yet been answered (nor addressed to) in the literature.

In the present work, we pursue an investigation on this theme, applying Painlevé's integrability test to a specific example, chosen due to a field-theoretical motivation: the most general forth-order supersymmetric potential with two bosonic variables, (19). Since the direct application of the test to the original supersymmetric potential reveals itself unfeasible, another symmetry, valid in many physical cases, was also required: parity. The analysis allowed us to conclude that, if parity is required to be valid only for the bosonic sector, then SUSY picks up just a very little number (see Table 3) of all the known integrable cases (shown in Tables 1 and 2) of a general (non-supersymmetric) fourth-order potential. If parity symmetry is implemented in a stronger version, such as to be also valid for the fermionic sector, then the integrable cases picked up by SUSY are, of course, even more restrictive (see Table 3).

Probably, the best way to estimate the effect of SUSY on integrability would be to compare the action of SUSY in reducing both the subsets in parameter space: the integrable and the non-integrable ones. Such a quantitative measure is difficult, but an intuitive candidate is the dimension of the subspaces that compose such subsets. Thus, let us compare the reduction induced by SUSY on the dimension of each one of these two subsets. The starting point is a generic fourth-order potential with x-parity symmetry, (4). The parameter space has dimension 7, while the integrable (known) sector is made up of the separable cases (dimension 5) and the various subspaces shown in Tables 1 and 2; the largest dimension of the corresponding non-separable cases is 3.

The effect of SUSY is the following: the parameter space is reduced to the five cases mentioned in Sect. 4.1; they make SUSY and x-parity compatible with each other. The largest dimension of the subsets that correspond to these five cases is 4 (the 2nd case). So, the overall reduction implemented by SUSY in parameter space is from dimension 7 to 4 (not taking into account the other subspaces with dimensions less than 4 that constitute the other four cases). Now, what is the action of SUSY separately on each of the integrable and non-integrable sectors? As far as Painlevé test could afford, Table 3 shows that the largest dimensionality of the subsets that constitute the integrable sector is 3. So, SUSY yields a reduction from 5 to 3 in the separable integrable cases. As for the non-separable cases, we can similarly notice a reduction from 3 to 2.

All these conclusions are summarized in Table 4. The lack of a precise criterium for deciding if the reduction in the parameter space induced by SUSY implementation is biased towards any of the two sectors (the integrable—separable or not—and the non-integrable) does not allow a final conclusion. However, independently of considering an absolute or a

Table 4 The action of SUSY on reducing the dimensionality of each sector of parameter space. The integrable sector (especially the non-separable subset) is the less affected

	x-parity symmetry valid	
	Without SUSY	With SUSY
Parameter space (whole)	Dim = 7	Dim = 4
	(see (4))	(The "5 cases")
Integrable, separable sector	Dim = 5	Dim = 3
	(see (4), $C_5 = C_8 = 0$)	(see (29))
Integrable, nonseparable sector	Dim = 3	Dim = 2
	Tables 1(b), 2(b), 2(d)	(Pot_6, see (39))
Non-integrable sector	$\mathbb{R}^7 - \{\dim = 5\}$	$\mathbb{R}^4 - \{\dim = 3\}$

relative variation in the dimensions, and, above all, considering that only the example of the present paper is being taken into account, SUSY seems to be less severe in the non-separable integrable cases and much more severe in the non-integrable sector. Anyway, the purpose of our work is just to point out some possibilities and to open the way towards the study of the possible relationships between SUSY and integrability, which is necessary for any definitive conclusion: to consider more examples, to study chaos implications (in our example and others), besides, of course and if possible, to invoke general arguments regarding the geometry and the topology of the sectors of parameter space, probably making use of the SUSY algebra and the theory of integrable systems.

Now, as for the (simple) chaos analysis carried out here, the main conclusion is that, regardless a probable tendency of SUSY to favor integrability (as seen above), a chaotic dynamics may indeed occur in a supersymmetric mechanical system with two bosonic degrees of freedom. Of course, more work is necessary to get the details.

Another good issue to be studied is the possible relationships (if any) between SUSY and chaos, that is, to study whether SUSY favors either chaos or regularity (or none).

Finally, we mention another issue that may be considered in subsequent works: the integrability analysis for the Grassmannian coordinates, ψ_i and $\bar{\psi}_i$ ($i = 1, 2$). Since we are dealing with two bosonic degrees of freedom (x and y), that is not a simple matter (the solution for the case of just one bosonic degree of freedom is presented in Ref. [35]). The reader may find interesting material about this (but restricted to the quantum case) in Refs. [40–42] (or, alternatively to the two latter ones, Sect. 2 of Ref. [30]). The difficulty is due to the quantity of Grassmannian coordinates (four), in contrast to the case of only two (as occurs with one-dimensional Witten's supersymmetric quantum mechanics [35] and with a planar quantum particle in an electromagnetic field [43]), whose integrability of the fermionic degrees of freedom is well established. Anyhow, a simple, although not conclusive argument is possible[1]: since, in a Hamiltonian system, the dynamics of a coordinate is given by its Poisson bracket with the Hamiltonian and, moreover, the SUSY charges have vanishing Poisson bracket with the Hamiltonian, then, the dynamics of a fermionic degree of freedom is given simply by the action of the SUSY charges on the Poisson bracket between its supersymmetric bosonic partner and the Hamiltonian. Thus, due to the time derivative present in the SUSY charges, the *tendency* of the fermionic degree of freedom is to show a *less* regular behavior than its bosonic partner. Of course, a more detailed analysis is required in order to

[1] We thank Dr. A.L.M.A. Nogueira for pointing this out.

draw more specific conclusions and find out possible exceptions to this reasoning (which do occur in field theory, as, for example, some special topological configurations such as BPS states).

Appendix Painlevé Test

The Painlevé test [38, 44] establishes if a system of ODEs exhibits the Painlevé property.

An ODE has the Painlevé property if its solutions in the complex plane are single-valued in the neighborhood of all its movable singularities. Given a differential system

$$L_j(u_i, u_{it}) = 0, \quad \text{with } i, j = 1, \dots, n, \tag{83}$$

we assume a Laurent expansion for the solution

$$u_i(t) = (t - t_0)^{\alpha_i} \sum_{k=0}^{\infty} u_{i,k}(t - t_0)^k, \tag{84}$$

with

$$u_{i,0} \neq 0 \quad \text{and} \quad \alpha_i \in Z^-, \tag{85}$$

where $u_{i,k}$ are constants. The algorithm for the Painlevé test is implemented by means of the following three steps:

Step 1 (Determine the leading singularity or dominant behavior).

We replace

$$u_i(t) \simeq u_{i,0}(t - t_0)^{\alpha_i} \tag{86}$$

into (83) to determine α_i and $u_{i,0}$ and we obtain an algebraic system with α_i, assuming negative integer values and t_0 arbitrary.

We require that two or more terms of each equation may balance and determine α_i and $u_{i,0}$.

If any α_i is not integer, the system is not of Painlevé type in its strong version.

If there are more than one solution for α_i or $u_{i,0}$ they define branches and the following steps of the algorithm need to be applied for each of these branches.

Step 2 (Determine the resonances).

For each α_i and $u_{i,0}$, we calculate the integers r for which $u_{i,r}$ is an arbitrary function in (83). We replace the truncated series

$$u_i(t) = u_{i,0}(t - t_0)^{\alpha_i} + u_{i,r}(t - t_0)^{\alpha_i + r} \tag{87}$$

in (83), and we look for integer r for which $u_{i,r}$ is an arbitrary constant.

To do that, after replacing the truncated series in (83), we keep the most singular terms in $(t - t_0)$, and the coefficients of $u_{i,r}$ are set to zero. We get:

$$Qu_r = 0, \quad u_r = (u_{1,r}\, u_{2,r} \cdots u_{M,r})^T, \tag{88}$$

with Q an $M \times M$ matrix depending on r.

The resonances are the roots of $\det(Q) = 0$.

In every system with the Painlevé property, the resonance (-1) will be present and corresponds to arbitrary $(t-t_0)$. The resonance with zero value may also be present, depending on the number of arbitrary values $u_{i,0}$.

Step 3 (Compatibility conditions and constants of motion).

For every resonance found in the previous step, there is a compatibility condition which must be verified in order that the system passes the Painlevé test. The compatibility conditions are verified by inserting

$$u_i(t) = (t-t_0)^{\alpha_i} \sum_{k=0}^{r_M} u_{i,k}(t-t_0)^k \tag{89}$$

into (83), where r_M is the highest positive integer resonance.

If all these compatibility conditions are satisfied so that they introduce a sufficient number of arbitrary constants, then the system is said to be of Painlevé type.

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